When Does a Developing Country Use New Technology?

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May 9, 2005

Abstract

We study an economy where capital accumulation and innovative activity take place within a two sector model. The first sector produces a consumption good using physical capital and non skilled labor according to a Cobb-Douglas production function. Technological progress in the production sector is driven by the research activity that takes place in the second sector. Research activity which produces new technologies requires technological capital and skilled labor along the line of a Cobb-Douglas production function. When new technologies produced by the research activity are used in the consumption sector they induce an endogenous increase of the Total Factor Productivity. The two kinds of capital are non substitutable while skilled and non skilled labors may or may not be substitutable.
We show that under conditions about the diffusion process of new technologies, the optimal strategy for a developing country is not to import technological capital at the first stage of development. In this case, all resources of the economy are devoted to consumption or investment in the physical capital and there is no research activity. This result is due to a threshold effect above which new technologies begin to have an impact on the productivity of the consumption sector. However, we show that there is some date after which it is optimal for the economy to import technological capital to produce new technologies. Our model exhibits an optimal pattern of economic development that is first rooted in physical capital accumulation and then by technical progress.

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1 Introduction

It is still rather unclear whether capital accumulation matters relatively more than technological progress in the growth of developing countries.
The Krugman – Solow controversy about the Asian Miracle illustrates particularly well this divergence between those who think that capital accumulation is an unimportant part of the growth process and those who think that it is a fundamental factor of growth for developing countries.

According to the traditional growth theories, poorer countries grow faster than richer ones during their first stage of development. That is due to decreasing returns to scale on capital accumulation which induce a catching-up process compatible with conditional convergence (Cass [1965]). However, cross-countries empirical studies show that development patterns differ considerably between countries in the long run (Barro&Sala-i-Martin [1995], Barro [1997]). These differences can be explained within a model of capital accumulation with convex – concave technology. In such a framework, Dechert and Nishimura [1983] prove the existence of threshold effect with poverty traps explaining "growth collapses" or taking-off. In the above models, the rate of growth is exogenous. Alternatively, other models propose several ways for endogeneizing the rate of growth. In that case, growth patterns are influenced by educational resources, innovation processes and technical differences between countries (see Romer 1986 for instance). Therefore, developing countries initially follow divergent paths of growth corresponding to the international differences in factor endowments. However, this point is in contradiction with most empirical studies on the role played by capital accumulation in early stages of growth. Notice that threshold effect is also used by Le Van and Saglam [2004] who show that a developing country can retain to invest in technology if the initial knowledge of the country and the quality of knowledge technology are low or if the level of fixed costs of the production technology are high.

In fact, it seems that the respective weights of capital accumulation and technical progress which may explain the rate of growth of a country depends on its level of development. The higher is the level of development the higher will be the weight of technical progress and the lower will be the scope of capital accumulation (Kim and Lau [1994])

If this idea is spreading, it seems there is still no model explaining the optimal switch of a country from one stage to another stage of development. In this paper, we present a model which may explain this switch. We show that the optimal pattern of growth of a developing country is initially determined by physical capital accumulation. Later, will appear technological progress when the country has reached a certain level of development. Capital accumulation and innovative activity take place within a two sector growth model. The first sector produces a consumption good using physical capital and non skilled labor according to a Cobb-Douglas production function. Technological progress in the production sector is driven by the research activity that takes place in the second sector. Research activity which produces new technologies requires technological capital and skilled labor along the line of a Cobb-Douglas production function. When new technologies produced by the research activity are used in the consumption sector they induce an endogenous increase of the Total Factor Productivity. The two kinds of capital are non substitutable while skilled and non skilled labors may or may not be substitutable.

We suppose that technological capital, used by the research activity, is not produced within the economy. The domestic economy must buy it in the international market at a given price. Consequently, the consumption good can be consumed, invested as physical capital or exported in order to import technological capital. The price of the consumption good is given by the international market and is used as numeraire in our economy. Physical capital is less costly than technological capital.

Our model exhibit first decreasing returns and then increasing returns. It differs from Dechert and Nishimura [1983] model which is convex-concave.

We show that under conditions about the diffusion process of new technologies, the optimal strategy for a developing country is not to import technological capital at the first stage of development. In this case, all resources of the economy are devoted to consumption or investment in the physical capital and there is no research activity. The growth process is initially driven by capital accumulation using concave technology. This result is due to a threshold effect above which new technologies begin to have an impact on the productivity of the consumption sector. Indeed, at early stages of development, new technologies cannot diffuse easily in the economy. This threshold is related to three factors: the level of available human capital, the price of technological capital and the initial income of the economy. Moreover, for given values of these factors, we show that there is some date after which it is optimal for the economy to import technological capital in order to produce new technologies. From that date onwards, research activity induces an endogenous technical change and the economy follows an optimal endogenous growth path with increasing returns to scale technology. Our model exhibits an optimal pattern of economic development that is first rooted in physical capital accumulation and then by technical progress. As a consequence of the existence of the threshold, a country might never take off and converges to a traditional steady state. This explains that the international convergence or divergence of income levels tightly depends on the value of the threshold effect.

The initial value of human capital plays an essential role in the process we have just described. The higher is this value the sooner research activity and endogenous growth will take place. This result is in aggreement with the empirical study of Benhabib and Spiegel [1994] showing that growth appears to be related to the initial level of human capital and not to the accumulation of human capital.

In the last part of the model (Section 5) we drop the assumption of non substitutability between the two kinds of labor. We allow high-skilled workers to work in the production sector but not the reverse. We show that the optimal endogenous growth path may be compatible with an underutilization of high-skilled labor in the research activity. But if the number of high-skilled workers is low relatively to the low-skilled ones then after some date the technological sector will completely use high-skilled workers.

Our paper is organized as follows. We set the model in Section 2. We first study the one period economy in Section 3. Section 4 deals with the infinitely lived optimal growth model with non substitutable labors. Finally, in Section 5, we allow high skilled workers mobility but not the reverse.

2 The Model

We consider a developping country which has a initial endowment X and produces a consumption good Y_d with a physical captial K_d and low-skilled labour L_d. This sector may use a quantity of new technology Y_e to increase its total productivity.

Formally we have

$$Y_d = \phi\left(Y_e\right) K_d^{\alpha_d} L_d^{1-\alpha_d}$$

where $\alpha_d \in (0,1)$ and ϕ is a non decreasing function which verifies $\phi(0) = x_0 > 0$.

The quantity of new technology Y_e may be produced through a Cobb-Douglas technology using technological capital K_e and high-skilled labor L_e. Formally we have:

$$Y_e = A_e K_e^{\alpha_e} L_e^{1-\alpha_e}.$$

This economy is not able to produce capital good. It exports its natural resourse X and imports capital goods. We use domestic capital good as numeraire. The prices are respectivily $\lambda > 1$ for technological capital and p_x for good X

The budget constraint of the economy is:

$$K_d + \lambda K_e = p_x X.$$

For simplicity, put $S = p_x X$ to get

$$K_d + \lambda K_e = S.$$

3 The One Period Model

The social planner will maximize

$$\max_{C, K_e, K_d, L_e, L_d} u(c)$$

under the constraints:

$$c = Y_d \tag{1}$$

$$Y_d = \phi(Y_e) K_d^{\alpha_d} L_d^{1-\alpha_d}, \tag{2}$$

$$Y_e = A_e K_e^{\alpha_e} L_e^{1-\alpha_e} \tag{3}$$

$$K_d + \lambda K_e = S \tag{4}$$

$$L_d \leq L_d^* \tag{5}$$

$$L_e \leq L_e^* h \tag{6}$$

and S is given.

The function u is strictly increasing. Thus, it is equivalent to maximize the quantity of consumption goods. In a one period model there will be no saving or investment. That explains (1). In (5) and (6), L_d^* and L_e^* are exogenous supplies of low-skilled and high-skilled workers. These inequalities assume there is no possible transfer between high-skilled and low-skilled workers. We suppose the human capital for high-skilled workers is measured by the number $h \geq 1$.

Let $\theta = \frac{\lambda K_e}{S}$. Then (4) is equivalent to

$$K_d = (1-\theta) S, \lambda K_e = \theta S. \tag{7}$$

Since at the optimum, $L_e = L_e^* h$, $L_d = L_d^*$, the social planner's problem turns out to be

$$\max_{\theta\in[0,1]} \phi\left(\frac{A_e h^{1-\alpha_e} {L_e^*}^{1-\alpha_e}}{\lambda^{\alpha_e}} \theta^{\alpha_e} S^{\alpha_e}\right)(1-\theta)^{\alpha_d} S^{\alpha_d} {L_d^*}^{1-\alpha_d} \tag{8}$$

Let $r_e = \frac{A_e}{\lambda^{\alpha_e}} {L_e^*}^{1-\alpha_e} h^{1-\alpha_e}$ and $\psi\left(r_e, S, \theta\right) = \phi\left(r_e \theta^{\alpha_e} S^{\alpha_e}\right)(1-\theta)^{\alpha_d} {L_d^*}^{1-\alpha_d}$
Solving the previous problem is equivalent to solve

$$\max_{0\leq\theta\leq 1} \psi\left(r_e, S, \theta\right) \tag{9}$$

In this program the function ϕ is important because it represents the diffusion of the research output in the economy. As we will see, in a fast diffusion situation, it is always interesting to use the technological capital. On the contrary, in a slow diffusion situation the developing country must be sufficiently wealthy (in its resource or its human capital) if they want to take off by buying technological capital.
Since the fonction ψ is continuous in θ, there will always be an optimal solution. Let

$$G(r_e, S) = Argmax\{\psi(r_e, S, \theta) : \theta \in [0,1]\}.$$

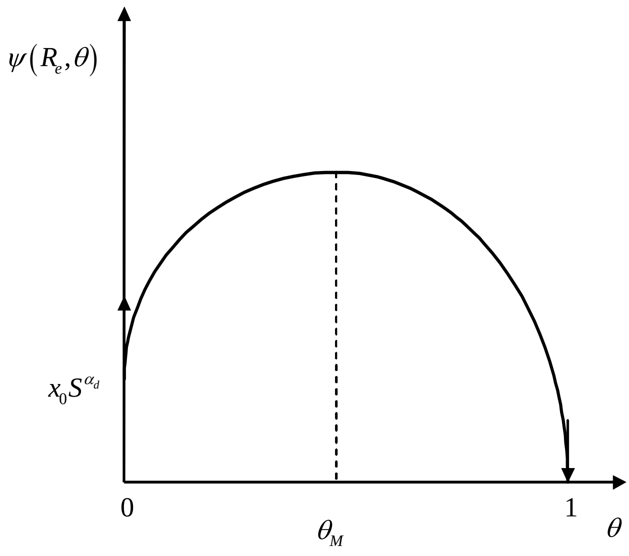
and $F(r_e, S) = \max\{\psi\left(r_e, S, \theta\right) : \theta \in [0,1]\}$. From the Maximum Theorem, F is continuous.

3.1 Case 1: fast diffusion of technology

Proposition 1 *Assume ϕ continuously differentiable, strictly increasing, $\phi'(0) > 0$. Then $G\left(r_e, S\right)$ is not empty. Moreover, $G\left(r_e, S\right) \subset \left]0,1\right[$*

Proof. It's obvious that the function ψ is continuouly differentiable in $\left]0,1\right[$. It satisfies $\psi'_\theta\left(r_e, S, 0\right) = +\infty$ and $\psi'_\theta\left(r_e, S, 1\right) = -\infty$. Hence, there exist θ_1 with $\psi(r_e, S, \theta_1) > \psi(r_e, S, 0)$ and θ_2 with $\psi(r_e, S, \theta_2) > \psi(r_e, S, 1)$. Therefore, the maximizer θ_M must be included in $\left]0,1\right[$. ■

Figure 1 gives the graph of $\psi\left(r_e, S, \theta\right)$ when ϕ strictly concave



3.2 Case 2: Slow diffusion of technology

In the previous case we suppose that any rise in the quantity of high technological capital, even very small, could have a direct effect. Now, we are going to analyze a slow diffusion of the technology created in the research sector. So, there will be a threshold effect. If the research is not sufficient to impose on the production sector, it will be useless.

The function ϕ is not strictly increasing, and supposed to have the following form:

$$\phi(x) = \left\{ \begin{array}{c} x_0, \forall x \leq X \\ x_0 + \gamma(x), \forall x \geq X \end{array} \right\}$$

with γ continuously differentiable, $\gamma' > 0, \gamma(X) = 0$.

Figure 2 gives the picture of the graph of ϕ when $\gamma(x) = ax$, $a > 0$

Figure 3 illustrates that the technological capital is useless since $r_eS^{\alpha_e} \leq X$

Figure 4 represent the case where $r_eS^{\alpha_e} > X$, $\bar{\theta}$ satisfy $r_eS^{\alpha_e}\bar{\theta} = X$

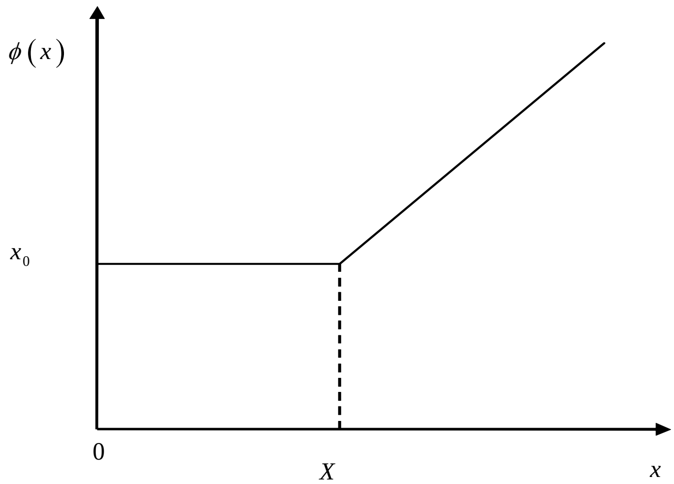


Figure 1:

Now we will prove that if S is high enough, then it will be efficient to use the technological capital, and when it is small, it is not efficient to import technological capital.

Proposition 2 *(i) If* $S \leq (\frac{X}{r_e})^{\frac{1}{\alpha_e}}$ *then* $G(r_e, S) = \{0\}$.
(ii) there exists $\hat{S}$ *such that:* $S > \hat{S} \Rightarrow G(r_e, S) \subset]0, 1[$.

Proof. (i) If $S \leq (\frac{X}{r_e})^{\frac{1}{\alpha_e}}$, then for any $\theta \in [0, 1]$, we have $r_e S^{\alpha_e} \theta^{\alpha_e} \leq X$, and hence $\psi(r, s, \theta) = x_0(1-\theta)^{\alpha_d} L_d^{*1-\alpha_d}$. Obviously, the maximizer is unique and equals 0.
(ii) Observe that $\forall S \geq 0, F(r_e, S) \geq x_0 L_d^{*1-\alpha_d}$. Let $S_0 > (\frac{X}{r_e})^{\frac{1}{\alpha_e}}$, and $\bar{\theta}(S_0)$ satisfy $r_e S_0^{\alpha_e} \bar{\theta}(S_0)^{\alpha_e} = X$. Let $\hat{\theta} \in]\bar{\theta}(S_0), 1[$. When S increases, $\bar{\theta}(S)$ decreases and $\hat{\theta}$ will be in $]\bar{\theta}(S), 1[$. We have $\psi(r_e, S, \hat{\theta}) \to +\infty$ when $S \to +\infty$. Hence, for S large enough, say, greater than some $\hat{S}$, then $\max_\theta\{\psi(r_e, S, \theta)\} > x_0 L_d^{*1-\alpha_d}$. That implies $0 \notin G(r_e, S)$. Since $\psi(r_e, S, 1) = 0$, we have $1 \notin G(r_e, S)$. The proof is complete. ■

Comment Observe that the larger is $r_e$the higher is the opportunity for the country to use new technology. Therefore, if the number of high-skilled

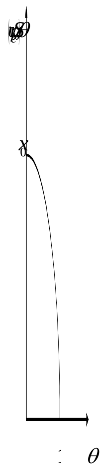


Figure 2:

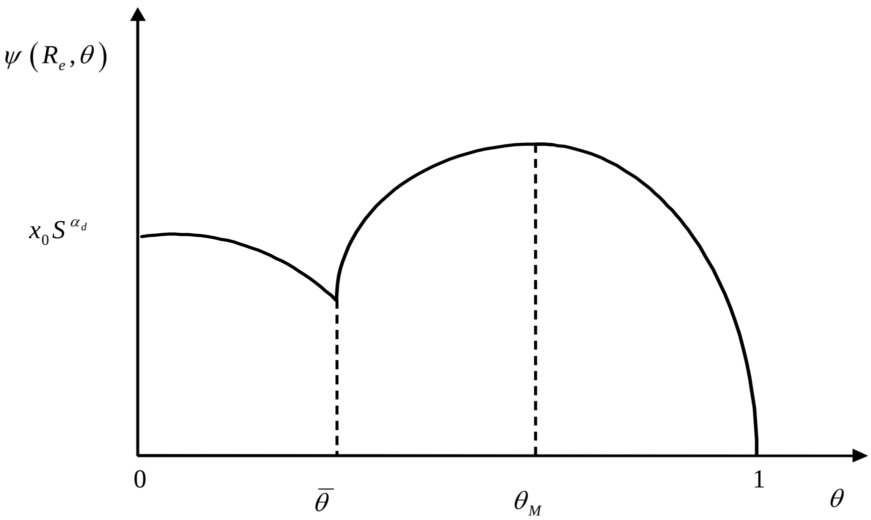


Figure 3:

workers is large, or their human capital is high, or if the price of new technology is low, or the productivity of the new technology production function is important, then the country will use new technology.

We want to prove there exists a critical value S^c, i.e., if $S < S^c$then $G(r_e, S) = \{0\}$and if $S > S^c$then $G(r_e, S) \subset]0, 1[$.

Let $B = \{S \geq 0 : F(r_e, S) = x_0 L_d^{*^{1-\alpha_d}}\}$.

Lemma 3 *B is non empty and compact.*

Proof. (i) B is not empty: obviously, $0 \in B$.
(ii) B is closed because the function F is continuous.
(iii)To prove that B is bounded: indeed, take a sequence S_n converging to $+\infty$. Fix some $\theta \in]0, 1[$. For n large enough, $\bar{\theta}(S_n) < \theta < 1$. Then $\psi(r_e, S_n, \theta)$ converges to $+\infty$. This implies $F(r_e, S_n) > x_0\ L_d^{*^{1-\alpha_d}}$for any n sufficiently large. That contradicts $S_n \in B$. ■

Proposition 4 *Let* $S^c = \max\{S : S \in B\}$. *Then if* $S < S^c$ *we have* $G(r_e, S) = \{0\}$ *and if* $S > S^c$ *then* $G(r_e, S) \subset]0, 1[$.

Proof. From Proposition 2(i), $S^c > 0$ since it must be greater than $(\frac{X}{r_e})^{\frac{1}{\alpha_e}}$. From Lemma 3, $S^c < +\infty$.
First, observe that if $S < S^c$ then $F(r_e, S) = x_0 L_d^{*^{1-\alpha_d}}$. Indeed, we have

$$\forall \theta \in [0, 1], \psi(r_e, S, \theta) \leq \psi(r_e, S^c, \theta).$$

Hence $F(r_e, S) = \max_\theta\{\psi(r_e, S, \theta)\} \leq \max_\theta\{\psi(r_e, S^c, \theta)\} = F(r_e, S^c) = x_0 L_d^{*^{1-\alpha_d}}$. Since $\forall S \geq 0, F(r_e, S) \geq x_0 L_d^{*^{1-\alpha_d}}$, we have $F(r_e, S) = x_0 L_d^{*^{1-\alpha_d}}$
Now, (i) if $S > S^c$, then from the very definition of S^c, we have $G(r_e, S) \subset]0, 1[$.
(ii) If $S < S^c$, then take some $S_0 < S^c$. Suppose for S_0 we have two solutions $\theta_M^1 = 0$ and $\theta_M^2 > 0$. There must be $\bar{\theta}_0 \in]0, 1[$ which satisfies $r_e S_0^{\alpha_e} (\bar{\theta}_0)^{\alpha_e} = X$ (if not, $\forall \theta, r_e S_0^{\alpha_e} \leq X$, and $G(r_e, S_0) = \{0\}$). For $\theta \in]0, \bar{\theta}_0]$, we have $\psi(r_e, S, \theta) = (1-\theta)^{\alpha_d} x_0 L_d^{*^{1-\alpha_d}} < \psi(r_e, S, 0) = x_0 L_d^{*^{1-\alpha_d}}$. Hence $\theta_M^2 > \bar{\theta}_0$. Let $S_0 < S_1 < S^c$, and $\bar{\theta}_1$ satisfy $r_e S_1^{\alpha_e} \bar{\theta}_1^{\alpha_e} = X$. Then $\theta_M^2 > \bar{\theta}_0 > \bar{\theta}_1$. We have

$$\phi(r_e S_0^{\alpha_e} (\theta_M^2)^{\alpha_e}) = x_0 + \gamma(r_e S_0^{\alpha_e} (\theta_M^2)^{\alpha_e})$$
$$\phi(r_e S_1^{\alpha_e} (\theta_M^2)^{\alpha_e}) = x_0 + \gamma(r_e S_1^{\alpha_e} (\theta_M^2)^{\alpha_e}) > \phi(r_e S_0^{\alpha_e} (\theta_M^2)^{\alpha_e}).$$

We obtain a contradiction

$$x_0 L_d^{*1-\alpha_d} = F(r_e, S_1) \geq \phi(r_e S_1^{\alpha_e} (\theta_M^2)^{\alpha_e})(1-\theta_M^2)^{\alpha_d} x_0 L_d^{*1-\alpha_d}$$

$$> \phi(r_e S_0^{\alpha_e} (\theta_M^2)^{\alpha_e})(1-\theta_M^2)^{\alpha_d} x_0 L_d^{*1-\alpha_d} = F(r_e, S_0) = x_0 L_d^{*1-\alpha_d}.$$

■

4 The Dynamic Model

4.1 The Model

We consider now an economy with one infinitely lived representative agent who has an intertemporal utility function. She has the possibilty to consume or to save at each period t. Savings are directly used to buy an equivalent amount of capital. This capital as before can be of two kinds, technological or production capital. As before, we suppose that the technological capital costs more than the production capital.There is no change in the production functions of the consumption goods and the new technology.

4.1.1 The program

The social planner will solve the following program.

$$\max \sum \beta^t u(c_t) \text{ with } 0 < \beta < 1,$$

under the constraints: for every date t,

$c_t + S_{t+1} \leq \phi(Y_{e,t}) K_{d,t}^{\alpha_d} L_{d,t}^{(1-\alpha_d)}$,
$Y_{e,t} = A_e K_{e,t}^{\alpha_e} L_{e,t}^{1-\alpha_e}$,
$L_{d,t} \leq L_d^*, L_{e,t} \leq h L_e^*$,
$K_{d,t} + \lambda K_{e,t} = S_t$.

The initial resource $S_0 > 0$ is given.

This problem is equivalent to:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

under the constraints: for every date t,

$c_t + S_{t+1} \leq H(r_e, S_t)$ with

$$H(r_e, S_t) = \max_{\theta} \phi(r_e \theta^{\alpha_e} S_t^{\alpha_e})(1-\theta)^{\alpha_d} L_d^{*1-\alpha_d} S_t^{\alpha_d},$$

and $S_0 > 0$ is given.
Recall that, as in the previous section, $r_e = \frac{A_e h^{1-\alpha_e} L_e^{1-\alpha_e}}{\lambda^{\alpha_e}}$.

We maintain the assumptions stated in Section 3 on ϕ. We add for this section:
(H_1) The function u is strictly concave, strictly increasing and satisfies $u(0) = 0$, the Inada Condition $u'(0) = +\infty$.

From the Maximum Theorem, H is continuous. It is obviously strictly increasing with respect to S and $H(r_e, 0) = 0$.
From Proposition 4, there exists a critical value S^c such that:
(i) If $S < S^c$, then the set of optimal values θ_M, $G(r_e, S)$, equals $\{0\}$, and
(ii) if $S > S^c$, then $G(r_e, S) \subset]0, 1[$.

Since the utility function is strictly increasing, at the optimum the constraints will be binding.

$$c_t = H(r_e, S_t) - S_{t+1}$$

A sequence $(S_t)_{t=0\ldots\infty}$ is called feasible from $S_0 \geq 0$ if we have $\forall t$, $0 \leq S_{t+1} \leq H(r_e, S_t)$. Thus the initial program is equivalent to the following problem

$$\max \sum_{t=0}^{\infty} \beta^t u(H(r_e, S_t) - S_{t+1})$$

under the constraints,

$$0 \leq S_{t+1} \leq H(r_e, S_t), \text{ for all } \ t \geq 0,$$

with $S_0 > 0$ given.

4.2 Properties of the optimal path

In this subsection, we will give some properties of the optimal path of our economy. In particular, we will show it is monotonic and does not converge to 0. Under some more stronger conditions, we will show that any optimal path with grow without bound. Along this growth path, after some date T, the economy will use new technology to produce consumption goods.

Lemma 5 *Every optimal path is monotonic*

Proof. Notice that we have the following Bellman equation. Let V be the value-function of the problem. We have

$$\forall S_0 \geq 0, V(S_0) = \max\{u(H(r_e, S_0) - S) + \beta V(S) : 0 \leq S \leq H(r_e, S_0)\}$$

Let Γ denote the optimal correspondence. From Amir [1996], this correspondence is non decreasing, i.e.,
if $S_0' < S_0$ then $\forall S_1' \in \Gamma(r_e, S_0')$, and $\forall S_1 \in \Gamma(r_e, S_0), S_1' \leq S_1$.
Hence, any optimal path must be monotonic. ■

Proposition 6 *Every optimal trajectory*(S_t^*) *from* $S_0 > 0$ *cannot converge to 0.*

Proof. Suppose that $S_t^* \to 0$.Then for $t \geq T$, we have: $S_t^* < S^c$. Hence, $\forall t > T, H(r_e, S_t^*) = x_0 L_d^{*1-\alpha_d} S_t^{*\alpha_d}$ and $H_S'(r_e, S_t^*) \to \infty$, because, $S_t \to 0$. As $u'(0) = +\infty$, we have Euler equation for $t > T$,

$$u'(c_t^*) = \beta u'\left(c_{t+1}^*\right) H_S'\left(r_e, S_{t+1}^*\right).$$

There exists $T_0 \geq T$ such that for all $t \geq T_0$ we have $H_S'\left(r_e, S_{t+1}^*\right)\beta > 1$. That implies $u'(c_t^*) > u'\left(c_{t+1}^*\right)$ or equivalently, $c_{t+1}^* > c_t^* \geq c_{T_0}^* > 0$. That is contradictory with $S_t^* \to 0$ (because it would have for consequence $c_t^* \to 0$). ■

Let us denote by $K_{d,t}^*$, $, K_{e,t}^*$ the optimal values of the capital stock for consumption good and the technological capital stock and let θ_t^* the associated optimal capital share, i.e. $K_{d,t}^* = (1-\theta_t^*)S_t^*$ and $\lambda K_{e,t}^* = \theta_t^* S_t^*$.

Proposition 7 *Let* S^s *be defined by* $x_0 L_d^{*1-\alpha_d} \alpha_d (S^s)^{\alpha_d - 1} = \frac{1}{\beta}$. *Suppose* $S^s > S^c$. *Let* $S_0 > 0$. *Then there exists* T *such that:* $\forall t > T, G(r_e, S_t^*) \subset]0,1[$ *or equivalently* $K_{e,t}^* > 0$.

Proof. When $S_0 \geq S^c$, since the optimal path (S_t^*) is nondeacresing, we have $\forall t > 0, G(r_e, S_t^*) \subset]0,1[$ or equivalently $K_{e,t}^* > 0$.
Consider the case $S_0 < S^c$. If for any t, $K_{e,t}^* = 0$, then the optimal path (S_t^*) will converge to S^s (see e.g. Le Van and Dana, [2005], Chapter 2). Since we assume $S^s > S^c$, there will be t with $S_t^* > S^c$. In this case $K_{e,t}^* > 0$ which is contradictory.
So, let T be the first date with $S_T^* > S^c$. Since the optimal path (S_t^*) is nondecreasing, we will have $K_{e,t}^* > 0$ for every $t > T$. ■
We will show that the optimal path S_t^* may converge to $+\infty$ with $\theta_t^* \to \theta^\infty = \frac{\alpha_e}{\alpha_e + \alpha_d}$. Thus, the optimal capital stocks converge also to $+\infty$.

We will now assume
(H_2) The function ϕ has the following form

$$\phi(x) = \left\{ \begin{array}{c} x_0, \forall x \leq X \\ x_0 + a(x - X), \forall x \geq X \text{ with } a > 0. \end{array} \right\}$$

Lemma 8 *Assume* (H_2)*. (i)The function* $F(r_e, S)$ *is continuously differentiable with respect to* S *in* $]0, S^c[\bigcup]S^c, +\infty[$*. At* S^c*, it has left derivative (equal to* 0*) and right derivative.*
(ii) For $S > S^c$*, there exists a unique* $\theta_M(S) \in G(r_e, S)$*. Moreover, when* S *converges to* $+\infty$*, then* $\theta_M(S)$ *converges to* $\theta^\infty = \frac{\alpha_e}{\alpha_e+\alpha_d}$*.*

Proof. (i) (a) When $S < S^c$, from Proposition 4, we have $F(r_e, S) = x_0 L_d^{*1-\alpha_d}$.
(b) Consider the case where $S > S^c$. Let $\bar{\theta}(S)$ satisfy $r_e S^{\alpha_e} \bar{\theta}(S)^{\alpha_e} = X$. Since $\psi(r_e, S, \theta) = x_0(1-\theta)^{\alpha_d} L_d^{*1-\alpha_d} \leq x_0 L_d^{*1-\alpha_d}$ when $\theta \leq \bar{\theta}(S)$, from the very definition of S^c, any solution must be larger than $\bar{\theta}(S)$. Thus, any solution θ must be interior to the interval $]\bar{\theta}(S), 1[$, because $\psi(r_e, S, 1) = 0$. The solution is unique since $\psi(r_e, S, \theta)$ is strongly concave in θ. One can check that $\frac{\partial\psi}{\partial\theta} < 0$. It satisfies $\psi'_\theta(r_e, S, \theta) = 0$. Tedious computations give:

$$\frac{\alpha_e}{\alpha_d}\theta^{\alpha_e-1}(1-\theta) = \frac{x_0 - aX}{a r_e S^{\alpha_e}} + \theta^{\alpha_e} \tag{10}$$

The left side member is a decreasing function in θ while the right side one is increasing in θ. The solution θ_M is unique. One can check that

$$\frac{d\theta_M}{dS} = \frac{A}{B} \tag{11}$$

with $A = \frac{aX - x_0}{a r_e} S^{\alpha_e - 1}$ and $B = \frac{\alpha_e - 1}{\alpha_d}\theta_M^{\alpha_e-2} - (1+\frac{\alpha_e}{\alpha_d})\theta_M^{\alpha_e-1} < 0$. Thus $F(r_e, \cdot)$ is differentiable for $S > S^c$.
(c) When $S = S^c$, there is a solution $\theta_M^1 = 0$ and another θ_M^2 which is the unique solution to equation (10). From Clarke ([1983] , theorem 2.8.2), there is a right derivative equal to $\psi'_S(r_e, S^c, \theta_M^2)$ and a left derivative which is trivially zero.
(ii) In (i) (b), we have shown that $G(r_e, S)$ is a singleton $\{\theta_M(s)\}$ when $S > S^c$. Taking the limit when $S \to +\infty$, we obtain that $\theta_M(S) \to \theta^\infty = \frac{\alpha_e}{\alpha_e+\alpha_d}$.
■

Proposition 9 *We maintain the same assumptions as in Lemma 8. We add*
(H_3) $\alpha_e + \alpha_d > 1$.
There exist $\bar{a}$ *and* $\bar{A}_e > 0$ *(or* $\bar{h}$*) such that if* $a = \bar{a}$*,* $A_e > \bar{A}_e$ *(or* $h > \bar{h}$*) then* $K_{e,t}^* \to +\infty$ *and* $\theta_t^* \to \theta^\infty = \frac{\alpha_e}{\alpha_e+\alpha_d}$*.*

Proof. It suffices to show that $H'_S(r_e, S) \neq \frac{1}{\beta}, \forall S$. Choose a large enough to have $x_0 - aX < 0$. In this case $\frac{d\theta_M(S)}{dS} < 0$ when $S \geq S_c$ (see (11)). Therefore, for every $S \geq S_c$, $\theta_M(S) > \theta^\infty$.

Let $\theta_M(S^c)$ be the unique maximizer associated with $F(r_e, S^c)$ which is strictly positive. For short, write θ_c instead of $\theta_M(S^c)$. Then (θ_c, S^c) satisfy (10) and $F(r_e, S^c) = x_0 {L^*}^{1-\alpha_d}$. We obtain:

$$(x_0 - aX)\frac{\alpha_e \theta_c^{\alpha_e - 1}(1-\theta_c)^{\alpha_d+1}}{(\alpha_e+\alpha_d)\theta_c^{\alpha_e} - \alpha_e\theta_c^{\alpha_e-1}} = x_0.$$

We see that θ_c is independent of r_e hence A_e and h. If A_e (or h) is large enough, then from (10) $S^c < S^s = (\beta x_0 \alpha_d)^{\frac{1}{1-\alpha_d}}$ and $H'_S(r_e, S) \neq \frac{1}{\beta}, \forall S \leq S^c$. Consider the case where $S > S^c$. From the envelope theorem and relation 10, we have

$$H'_S(r_e, S) = a r_e \alpha_e \theta_M(S)^{\alpha_e - 1}(1-\theta_M(S))^{\alpha_d} S^{\alpha_e+\alpha_d-1}.$$

We can bound this derivative from below:

$$H'_S(r_e, S) > a r_e \alpha_e (1-\theta_c)^{\alpha_d} S^{\alpha_e+\alpha_d-1}.$$

and hence

$$S < (\frac{1}{\beta a r_e \alpha_e (1-\theta_c)^{\alpha_d}})^{\frac{1}{\alpha_e+\alpha_d-1}}.$$

Again from (10) we can write $S^c = (\frac{\zeta(a,x_0,X))}{a r_e})^{\frac{1}{\alpha_e}}$ where the function ζ can be easily computed. One can also easily check that if A_e (hence r_e) is sufficiently large then S will be less than S^c which is a contradiction. ■

Remark 10 *Actually, we have also shown that S^c is a decreasing function of A_e and h.*

5 Mobility of labour

We now assume that high-skilled people can work in the sector of consumption good if the demand for high-skilled labor is not sufficient. But the reverse is not possible, i.e. low-skilled people cannot move in the new technology sector. We therefore replace the constraints labor demands (5,6) by

$$L_d \leq L_e^* + L_d^* \tag{12}$$

and

$$L_e \leq h L_e^* \tag{13}$$

We can write $L_e = \mu h L_e^*, L_d = L_d^* + (1-\mu) L_e^*$ with $\mu \in [0,1]$. We assume that when the high-skilled workers are in the consumption sector their human capital equals the one of this sector.
The production function in the new technology sector will be

$$Y_e = \frac{A_e}{\lambda^{\alpha_e}} \theta^{\alpha_e} S^{\alpha_e} \mu^{1-\alpha_e} h^{1-\alpha_e} {L_e^*}^{1-\alpha_e}$$

where μ represents the part of high skill labor used in this sector.
The production function in the consumption good sector now is:

$$Y_d = \phi(Y_e)(1-\theta)^{\alpha_d} S^{\alpha_d} \left(L_d^* + (1-\mu) L_e^*\right)^{1-\alpha_d}$$

5.1 The one period model

Let $r_e = \frac{A_e}{\lambda^{\alpha_e}} h^{1-\alpha_e} {L_e^*}^{1-\alpha_e}$. The program of the social planner is:

$$\max_{\substack{0 \le \theta \le 1 \\ 0 \le \mu \le 1}} \quad Y_d = \phi\left(r_e \theta^{\alpha_e} S^{\alpha_e} \mu^{1-\alpha_e}\right)(1-\theta)^{\alpha_d} S^{\alpha_d} \left(L_d^* + (1-\mu) L_e^*\right)^{1-\alpha_d}.$$

Let

$$G(r_e, S, \theta, \mu) = \phi\left(r_e \theta^{\alpha_e} S^{\alpha_e} \mu^{1-\alpha_e}\right)(1-\theta)^{\alpha_d} \left(L_d^* + (1-\mu) L_e^*\right)^{1-\alpha_d}.$$

The problem is equivalent to

$$\max_{(\theta,\mu)\in[0,1]\times[0,1]} G(r_e, S, \theta, \mu).$$

Let

$$F(r_e, S) = \max_{(\theta,\mu)\in[0,1]\times[0,1]} G(r_e, S, \theta, \mu).$$

Then $F(r_e, S) \ge x_0 \left(L_d^* + L_e^*\right)^{1-\alpha_d}$. As before, define $B = \{S \ge 0 : F(r_e, S) = x_0 \left(L_d^* + L_e^*\right)^{1-\alpha_d}\}$. It is easy to check that B is compact and nonempty. The critical value is

$$S^c = \max\{S : S \in B\}$$

Observe that for $S > S^c$ the function

$$Z(r_e, S, \theta, \mu) = Log(G(_r e, S, \theta, \mu))$$

is strongly concavev in (θ, μ). Since to maximize $G(_r e, S, \theta, \mu)$ is equivalent to maximize $Z(r_e, S, \theta, \mu)$ when $S > S^c$, the solution $(\theta_M(S), \mu_M(S))$ will be unique. Obviously, if $S > S^c$ then $\theta_M(S) > 0$ (if not, we will have $\mu_M(S) = 0$ and $F(r_e, S) = x_0 \left(L_d^* + L_e^*\right)^{1-\alpha_d}$). We have the following result

Proposition 11 *Assume* $\frac{L_e^*}{L_d^*} < \frac{1-\alpha_e}{1-\alpha_d}$. *Then there exists* $\bar{S}$ *such that if* $S > \bar{S}$ *then* $\mu_M(S) = 1$.

Proof. Assume the statement false. Then there exists a sequence (S_n) converging to $+\infty$ with $\mu_M(S_n) < 1, \forall n$. We may assume $\mu_M(S_n) \to \bar{\mu} \leq 1$ and $\theta_M(S_n) \to \bar{\theta}$. For short, write $\mu_n = \mu_M(S_n)$, $\theta_n = \theta_M(S_n)$. For every n, we have

$$x_0 + a(r_e S_n^{\alpha_e} \theta_n^{\alpha_e} \mu_n^{1-\alpha_e} - X)(1-\theta_n)^{\alpha_d}(L_d^* + (1-\mu_n)L_e^*)^{1-\alpha_d}$$

$$\geq x_0 + a(r_e S_n^{\alpha_e} \theta^{\alpha_e} \mu^{1-\alpha_e} - X)(1-\theta)^{\alpha_d}(L_d^* + (1-\mu)L_e^*)^{1-\alpha_d}$$

for every $\theta \in [0,1]$, every $\mu \in [0,1]$. This inequality is equivalent to

$$\frac{x_0}{S_n^{\alpha_e}} + a(r_e \theta_n^{\alpha_e} \mu_n^{1-\alpha_e} - \frac{X}{S_n^{\alpha_e}})(1-\theta_n)^{\alpha_d}(L_d^* + (1-\mu_n)L_e^*)^{1-\alpha_d}$$

$$\geq \frac{x_0}{S_n^{\alpha_e}} + a(r_e \theta^{\alpha_e} \mu^{1-\alpha_e} - \frac{X}{S_n^{\alpha_e}})(1-\theta)^{\alpha_d}(L_d^* + (1-\mu)L_e^*)^{1-\alpha_d}.$$

Let S_n converge to infinity. We obtain

$$a r_e \bar{\theta}^{\alpha_e} \bar{\mu}^{1-\alpha_e}(1-\bar{\theta})^{\alpha_d}(L_d^* + (1-\bar{\mu})L_e^*)^{1-\alpha_d}$$

$$\geq a r_e \theta^{\alpha_e} \mu^{1-\alpha_e}(1-\theta)^{\alpha_d}(L_d^* + (1-\mu)L_e^*)^{1-\alpha_d} > 0 \text{ if } \theta \in]0,1[, \mu > 0.$$

That implies $\bar{\theta} \in]0,1[, \bar{\mu} > 0$. But for every n we also have:

$$x_0 + a(r_e S_n^{\alpha_e} \theta_n^{\alpha_e} \mu_n^{1-\alpha_e} - X)(1-\theta_n)^{\alpha_d}(L_d^* + (1-\mu_n)L_e^*)^{1-\alpha_d}$$

$$\geq x_0 + a(r_e S_n^{\alpha_e} \theta_n^{\alpha_e} \mu^{1-\alpha_e} - X)(1-\theta_n)^{\alpha_d}(L_d^* + (1-\mu)L_e^*)^{1-\alpha_d}.$$

Since $\mu_n \in (0,1)$ we get the first order condition for μ_n:

$$a r_e \theta_n^{\alpha_e}(1-\alpha_e)\mu_n^{-\alpha_e}(L_d^* + (1-\mu)L_e^*) =$$

$$L_e^*(1-\alpha_d)[\frac{x_0 - aX}{S_n} + a r_e \theta_n^{\alpha_e} \mu_n^{1-\alpha_e}].$$

Let S_n converge to infinity. We obtain $\bar{\mu} = \frac{(1-\alpha_e)(L_d^* + L_e^*)}{(2-\alpha_e-\alpha_d)L_e^*}$. And $\bar{\mu} > 1$ if $\frac{L_e^*}{L_d^*} < \frac{1-\alpha_e}{1-\alpha_d}$. That implies $\mu_n = 1$ for any n large enough. ■

Proposition 12 *Let* $S > S^c$. *Assume* $\frac{L_e^*}{L_d^*} \leq \frac{1-\alpha_e}{1-\alpha_d}$ *and* $x_0 - aX < 0$. *Then* $\mu_M(S) = 1$.

Proof. To make short, write (θ_M, μ_M) instead of $(\theta_M(S), \mu_M(S))$. If (θ_M, μ_M) are interior we have the following first-order conditions

$$\theta_M^{\alpha_e - 1}\mu_M^{1-\alpha_e}\alpha_e(1-\theta_M) = \left[\frac{x_0 - aX}{ar_e S^{\alpha_e}} + \theta_M^{\alpha_e}\mu_M^{1-\alpha_e}\right]\alpha_D, \tag{14}$$

$$\theta_M^{\alpha_e}\mu_M^{-\alpha_e}(1-\alpha_e)(L_d^* + (1-\mu_M)L_e^*) = \left[\frac{x_0 - aX}{ar_e S^{\alpha_e}} + \theta_M^{\alpha_e}\mu_M^{1-\alpha_e}\right](1-\alpha_d)L_e^*. \tag{15}$$

If $x_0 - aX < 0$, then from (15)we obtain $\mu_M > \mu^\infty = \frac{(1-\alpha_e)(L_d^*+L_e^*)}{(2-\alpha_d-\alpha_e)L_e^*}$. But if $\frac{L_e^*}{L_d^*} \leq \frac{1-\alpha_e}{1-\alpha_d}$, then $\mu^\infty \geq 1$ and we have a contradiction. Since $\mu_M > 0$, we must have $\mu_M = 1$. ■

Define

$$L(r_e, S) = F(r_e, S)S^{\alpha_d}.$$

The function L is strictly increasing in S, continuous, $L(r_e, 0) = 0$.

5.2 The Dynamic Model

As before the optimal growth model is:

$$\max_{\substack{0 \leq S_{t+1} \leq L(r_e, S_t) \\ S_0 > 0 \text{ given}}} \sum \beta^t u\left(L\left(r_e, S_t\right) - S_{t+1}\right)$$

Since the function $F(r_e, \cdot)$ is strictly increasing, the optimal path (S_t^*) is monotonic.

We have the following proposition

Proposition 13 *Let S^s satisfy*

$$\alpha_d(S^s)^{\alpha_d - 1}x_0(L_e^* + L_d^*)^{1-\alpha_d} = \frac{1}{\beta}.$$

Assume $S^s > S^c$. Then any optimal path (S_t^*) from $S_0 > 0$ cannot converge to zero. Moreover, there will be a date T such that , for all $t > T$, the optimal technological capital stock satisfies $K_{e,t}^* > 0$.

Proof. If the optimal path (S_t^*) converges to zero, then for all t large enough, we have $S_t^* < S_c$. That implies the optimal values θ_t^* and μ_t^* are respectively 0 and 1. To end the proof, see the proofs of Propositions 6, 7. ■

Observe that along an optimal path, we cannot ensure the supply of high-skilled labor supply be exhausted. The following proposition gives a condition for which that will be true.

Proposition 14 *Assume that the optimal path* (S_t^*) *converges to* $+\infty$ *and* $\frac{L_e^*}{L_d^*} < \frac{1-\alpha_e}{1-\alpha_d}$. *Then there exists* T *such that* $\forall t > T, \mu_t^* = 1$.

Proof. That is a corollary of Proposition 11 ■

Finally we have the following proposition.

Proposition 15 *Assume* $S^c < S^s$, $\frac{L_e^*}{L_d^*} < \frac{1-\alpha_e}{1-\alpha_d}$ *and* $x_0 - aX < 0$. *Then (i) for any* $S_0 >$, *we have (a)* $\mu_t^* = 0$ *if* $S_t^* \leq S^c$ *and* $\mu_t^* = 1$ *when* $S_t^* > S^c$.
(ii) Moreover, if Assumption (H_3) *holds, then there exist* $\bar{a}$ *and* $\bar{A}_e > 0$ *(or* $\bar{h}$*) such that if* $a = \bar{a}$, $A_e > \bar{A}_e$ *(or* $h > \bar{h}$*) then* $K_{e,t}^* \to +\infty$ *and* $\theta_t^* \to \theta^\infty = \frac{\alpha_e}{\alpha_e+\alpha_d}$.

Proof. (i) When $S_0 > S^c$, use Proposition 12 to obtain the result. When $S_0 \leq S^c$, there will be a date T (which is the first) where the optimal $S_t^* \geq S^c$ since $S^c < S^s$. For $t \leq T$, we have $H(r_e, S_t^*) = x_0(L_d^* + L_e^*)^{1-\alpha_d} S_t^{*\alpha_d}$ and $\mu_t^* = 0, \forall t \leq T$. When $S_t^* > S^c$, apply Proposition 12 to get $\mu_t^* = 1$.
(ii) The argument is similar to the one in Proposition 9. ■

Remark 16 *Consider the case where* $x_0 = aX$ *and* $\alpha_e + \alpha_d > 1$. *From the first order conditions (14, 15) when* $S > S^c$, *we obtain that the optimal values* (θ^*, μ^*) *are independent of* S *and respectively equal*

$$\theta^\infty = \frac{\alpha_e}{\alpha_e + \alpha_d}, \ \mu^\infty = \frac{(1-\alpha_e)(L_d^* + L_e^*)}{(2 - \alpha_d - \alpha_e)L_e^*}.$$

The technological function of the optimal growth model becomes

$$L(r_e, S) = x_0(L_d^* + L_e^*)^{1-\alpha_d} S^{\alpha_d} \ \text{if} \ \ S \leq S^c,$$

and

$$L(r_e, S) = ar_e {\theta^\infty}^{\alpha_e} (1-\theta^\infty)^{\alpha_d} {\mu^\infty}^{1-\alpha_e} (L_d^* + (1-\mu^\infty)L_e^*)^{1-\alpha_d} S^{\alpha_e+\alpha_d-1}, \ \text{if} \ \ S > S^c.$$

The function $L(r_e, \cdot)$ *is concave-convex and differs from the case in Dechert and Nishimura (1983) where the technology is convex-concave.*

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