Quality of Knowledge Technology, Returns to Production Technology and Economic Development*

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Abstract

Presenting a discrete time version of the Romer (1986) model, this paper analyzes optimal paths in a one-sector growth model when the technology is not convex. We prove that for a given quality of knowledge technology, the countries could take-off if their initial stock of capital are above a critical level; otherwise they could face a poverty-trap. We show that even a developed country having a concave production function may face a poverty-trap if endowed with a low quality of knowledge technology.

Keywords : Optimal growth, optimal path, value function, poverty-trap, increasing returns.

1 Introduction

Convex structures of the technology and prefences have played an important role in economic analysis of optimal one-sector growth models. They guarantee that the sequence of optimal stocks moves monotonically towards a unique steady state. (as in Cass (1965) and Koopmans (1965)). In these models, per-capita output should converge to a steady state due to the assumption of diminishing returns to per-capita capital in the production of per-capita output. However these studies were unable to explain the non-convergence of countries whose potential causes could be the different time preferences, technologies, demographies, market structures or economic policies.

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In a model of endogenous technological change in which the knowledge accumulated by the agents is the basic form of capital, Romer (1986) relaxing this usual assumption of diminishing returns showed that per-capita output can grow without bound and the level of per-capita output accross different countries need not converge. In this analysis, new technology created by a single firm which has a positive external effect on the other firms is assumed to be the product of a research technology that exhibits diminishing returns. Thus, whereas production as a function of the firm exhibits diminishing returns, production as a function of the stock of knowledge in the economy is assumed to exhibit increasing returns.

On the other hand, Majumdar and Mitra (1982) and Dechert and Nishimura (1983) analyzed an optimal growth model with a non-convex technology. Their key result was that the sequence of capital stocks is necessarily monotonic and under some assumptions they exhibit a poverty trap. Extending the analysis to an open country, Askenazy and Le Van (1998) in a continous time framework and Dimaria and Le Van (2000) in a discrete time framework also showed that if the debt constraint is hard, it could be optimal for a poor country to collapse while a rich country to converge to a high level of steady state.

In this paper, following Dimaria, Morhaim and Le Van (2000), we present a discrete time version of the Romer (1986) model and relax a fundamental hypothesis: nonconcavity of the production function. We analyze the case of a developing country with a production technology that exhibits linear production-capital ratio at the early stages of industrialization. Then for higher capital stocks, the production function becomes concave as in the case of a developed country. We prove the existence of solutions to the social-planner problem and characterize the properties of the optimal paths. We show that a for a given quality of knowledge technology, the countries could take-off if their initial stock of capital are above a critical level; otherwise they could face a poverty-trap. We show that even a developed country having a concave production function may face a poverty-trap if endowed with a low quality of knowledge technology.

The paper is organized as follows. In Section 2, presenting the model and its assumptions, we study the existence of solutions to the social-planner problem and analyze the properties and convergence of optimal paths. Finally, Section 3 concludes.

2 The Infinite-Horizon Growth Model

We consider a closed economy in which the preferences of the S identical consumers are globally represented by a strictly concave utility function of consumption, $u(c)$.

(U1) $u(c)$ is twice continously differentiable, $u'(c) > 0$, $u''(c) < 0, \forall c > 0$ and $u(0) = 0$, $u'(0) = +\infty$.

The instantaneous production of output for a firm is given by $F(k_t, K_t, x_t)$, which depends on the firm specific knowledge (k_t), the aggregate knowledge (K_t), and the level of all other factors such as physical capital, labour, etc. To simplify and to have per-firm and per-capita values coincide, we restrict our attention to an equilibrium in which the number of firms and the number of consumers are equal by assuming that $S = N = 1$. Following Romer (1986) and Le Van, Morhaim and Dimaria (2000), we assume that the additional factors are fixed in supply so that the optimal solution for x is $\bar{x}$. Dropping $\bar{x}$ from the production function, let f, h and $\mathcal{F}$ be:

$$F(k, K, \bar{x}) = f(k)\ h(K)$$
$$\mathcal{F}(k) := F(k, k, \bar{x}) = f(k)\ h(k).$$

We consider two cases:
i) a developed country where the production function is concave:

$$\begin{aligned} f(k) &= k^{\mu}, \\ h(k) &= k^{\rho},\ \rho > 0 \end{aligned}$$

ii) a developing country where the production function is a linear function in an initial phase and concave afterwards:

$$\begin{aligned} f(k) &= \begin{cases} \delta k, & k \leq \bar{k} \\ A + k^{\mu}, & k \geq \bar{k} \end{cases} \quad \textit{and} \\ h(k) &= k^{\rho},\ \rho > 0 \end{aligned}$$

with $\delta - \mu \bar{k}^{\mu-1} < 0$, $0 < \delta < 1$, $0 < \mu < 1$ and $1 < \mu + \rho$. Note that $A < 0$ as $A = \delta \bar{k} - \bar{k}^{\mu} = \bar{k}\left(\delta - \bar{k}^{\mu-1}\right) < \bar{k}\left(\mu \bar{k}^{\mu-1} - \bar{k}^{\mu-1}\right) = \bar{k}^{\mu}(\mu - 1) < 0.$

Investing an amount I_t of forgoing consumption, a firm with a current stock of private knowledge k_t produces additional knowledge which induces a rate of growth

$$k_{t+1} - k_t = G(I_t, k_t)$$

Assume that:
(G1) G is concave and homogenous of degree one. Then:

$$\frac{k_{t+1} - k_t}{k_t} = G(\frac{I_t}{k_t}, 1) = g(\frac{I_t}{k_t})$$

(G2) $g(0) = 0$, $g'(0) = \frac{1}{\lambda} < +\infty$
(G3) $0 \leq g(y) \leq \alpha$.

For an arbitrary path K, the social optimization problem maximizes the utility of a representative consumer subject to the technology implied by the path K.

$$Maximize \sum_{t=0}^{+\infty} \beta^t u(c_t)$$
$$s.t. \quad 0 \leq \frac{k_{t+1} - k_t}{k_t} \leq g(\frac{\mathcal{F}(k_t) - c_t}{k_t})$$
$$k_0 > 0, given.$$

Note that in social optimization problem, the production function exhibits an initial phase of increasing returns and a second phase with decreasing returns. Note also that this problem is equivalent to:

$$Maximize \sum_{t=0}^{+\infty} \beta^t u(\mathcal{F}(k_t) - k_t \gamma(\frac{k_{t+1} - k_t}{k_t}))$$
$$s.t. \ k_t \leq k_{t+1} \leq k_t g(\frac{\mathcal{F}(k_t)}{k_t}) + k_t$$
$$k_0 > 0, given.$$

where $\gamma := g^{-1}$.

Following from **(G3),** note that $k_t \leq k_{t+1} \leq k_t(1+\alpha)$. Then we have, $k_t \leq k_0(1+\alpha)^t$ inducing that, $\mathcal{F}(k_t) \leq \mathcal{F}(k_0(1+\alpha)^t) \leq \beta \ k_0^{\rho+1} \left[(1+\alpha)^{(\rho+1)}\right]^t$. In what follows we assume that:

(P1) $\quad 0 < \beta < 1$ and $\beta(1+\alpha)^{(\rho+1)} < 1$.

With $0 \leq k_t \leq k_{t+1} \leq \bar{k}$, we have $\gamma(\frac{k_{t+1}-k_t}{k_t}) \leq \frac{\mathcal{F}(k_t)}{k_t} \leq \frac{\mathcal{F}(\bar{k})}{\bar{k}}$ so that:

$$\left(\frac{k_{t+1}}{k_t} - 1\right) \leq g\left(\frac{\mathcal{F}(\bar{k})}{\bar{k}}\right) = \bar{g} < \alpha.$$

Assume that:

(P2) $\quad \gamma(x) = \lambda x, \ 0 \leq x \leq \bar{g}$.

Note that, at zero a marginal increase in the amount of investment in terms of forgoing consumption induces an increase in the stock of private knowledge less than $+\infty$ which enables us to interpret that λ reflects the quality of the knowledge technology.

2.1 Existence of a Solution

A sequence $\tilde{k} = (k_t)_t$ is called feasible from k_0 if it satisfies the constraints of the social optimization problem:

$$\forall t, k_0 \leq k_{t+1} \leq k_t g\left(\frac{\mathcal{F}(k_t)}{k_t}\right) + k_t$$

In this section, we first prove that every feasible sequence from k_0 belongs to a compact set for the product topology; second we show that the objective function is continous for this topology. Existence of solutions follows from these results.

2.1.1 Compactness

Let $\tilde{k}$ be a feasible path from k_0. Then by assumption (G3), for every t:

$$k_t \leq k_{t+1} \leq k_t g(\frac{\mathcal{F}(k_t)}{k_t}) + k_t \leq (1+\alpha)k_t$$

so, $k_t \leq (1+\alpha)^t k_0$, that is:

$$\tilde{k} \in \prod_{t=0}^{+\infty} \left[k_0, (1+\alpha)^t k_0\right].$$

Thus, $\tilde{k}$ belongs to a compact set for the topology. Since g and $\mathcal{F}$ are continous, the feasible set from k_0 is compact for the product topology.

2.1.2 Continuity of the objective function

The objective function is:

$$\mathcal{U} : \tilde{k} \longrightarrow \sum_{t=0}^{+\infty} \beta^t u\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right).$$

We want to show that $\mathcal{U}$ is continous for the product topology. Let $\tilde{k}^n$ be a feasible path from k_0 that converges to $\tilde{k}$. First note that,

$$\begin{aligned}
&\left|\sum_{t=0}^{+\infty} \beta^t u\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right) - \sum_{t=0}^{+\infty} \beta^t u\left(\mathcal{F}(k_t^n) - k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right| \\
&\leq \sum_{t=0}^{+\infty} \beta^t \left|u\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right) - u\left(\mathcal{F}(k_t^n) - k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right| \\
&= \sum_{t=0}^{T-1} \beta^t \left|u\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right) - u\left(\mathcal{F}(k_t^n) - k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right| + \\
&\sum_{t=T}^{+\infty} \beta^t \left|u\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right) - u\left(\mathcal{F}(k_t^n) - k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right|.
\end{aligned}$$

Moreover,

$$\sum_{t=T}^{+\infty}\beta^t\left|u\left(\mathcal{F}(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)-u\left(\mathcal{F}(k_t^n)-k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right|\leq$$

$$\sum_{t=T}^{+\infty}\beta^t u\left(\mathcal{F}(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)+\sum_{t=T}^{+\infty}\beta^t u\left(\mathcal{F}(k_t^n)-k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\leq$$

$$\sum_{t=T}^{+\infty}\beta^t u\left(\mathcal{F}(k_t)\right)+\sum_{t=T}^{+\infty}\beta^t u\left(\mathcal{F}(k_t^n)\right),\quad \text{as } u \text{ is strictly increasing.}$$

But also, as u is concave and $k_t \leq k_0(1+\alpha)^t$:

$$\begin{aligned} u\left(\mathcal{F}(k_t)\right) &\leq u\left(\mathcal{F}\left((1+\alpha)^t k_0\right)\right)\leq u\left(\beta\,(1+\alpha)^{(\rho+1)t}\,k_0^{\rho+1}\right)\\ &\leq \beta\,(1+\alpha)^{(\rho+1)t}\,u\left(k_0^{\rho+1}\right). \end{aligned}$$

Thus,

$$\sum_{t=T}^{+\infty}\beta^t\left|u\left(\mathcal{F}(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)-u\left(\mathcal{F}(k_t^n)-k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right|\leq$$

$$2\beta u\left(k_0^{\rho+1}\right)\sum_{t=T}^{+\infty}\left[\beta\,(1+\alpha)^{\rho+1}\right]^t.$$

By means of assumption (P1), there exists T_1 such that:

$$\sum_{t=T_1}^{+\infty}\left[\beta\,(1+\alpha)^{\rho+1}\right]^t\leq\frac{\varepsilon}{4\beta\;u\left(k_0^{\rho+1}\right)}$$

Then for $T=T_1$, one has:

$$\sum_{t=T}^{+\infty}\beta^t\left|u\left(\mathcal{F}(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)-u\left(\mathcal{F}(k_t^n)-k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right|\leq\frac{\varepsilon}{2}$$

$$\sum_{t=0}^{T-1}\beta^t\left|u\left(\mathcal{F}(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)-u\left(\mathcal{F}(k_t^n)-k_t^n\gamma\left(\frac{k_{t+1}^n-k_t^n}{k_t^n}\right)\right)\right|\leq\frac{\varepsilon}{2}$$

Thus, the objective function is continous and since the feasible path set is compact for the product topology, the problem has a solution.

2.2 Value function, Bellman equation

Let $V(k_0)$ denote the value function of the social optimization problem. It is clear that the value fuction verifies the Bellman equation:

$$V(k_0)=\max_{k_0\leq y\leq k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right)+k_0}\left\{u\left(\mathcal{F}(k_0)-k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right)+\beta V(y)\right\}$$

V is the unique continous solution to Bellman equation. Let

$$\varphi(k_0) = \underset{k_0 \le y \le k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right)+k_0}{\arg\max} \left\{ u\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right) + \beta V(y) \right\}.$$

By the maximum theorem, φ is upper semi-continous.

2.3 Properties and convergence of optimal paths

In this section we derive some properties of optimal paths. First we show the non-nullity of optimal consumption and capital. Second, we prove that there exists a critical value k_c which may be interpreted as the "top" of the poverty trap.

Lemma 1 *$\frac{\mathcal{F}(k)}{k}$ and $\mathcal{F}'(k)$ are increasing functions.*

Proof. *i)* $\left(\frac{\mathcal{F}(k)}{k}\right)' = \frac{1}{k^2}\left[\mathcal{F}'(k)k - \mathcal{F}(k)\right]$. *What follows from the definition of $\mathcal{F}(k)$ is that:*

$$\mathcal{F}'(k)k - \mathcal{F}(k) = \begin{cases} \delta\rho k^{\rho+1}, & k < \bar{k} \\ A(\rho-1)k^{\rho} + \mu k^{\mu+\rho} + (\rho-1)k^{\mu-\rho}, & k \ge \bar{k}. \end{cases}$$

There are two cases to be checked when $k \ge \bar{k}$:

$\rho \le 1 \Rightarrow \mathcal{F}'(k)k - \mathcal{F}(k) = k^{\rho}\left[A(\rho-1) + (\mu+\rho-1)k^{\mu}\right] > 0$ *as* $A = \delta\bar{k} - \bar{k}^{\mu} < 0$ *and*

$\rho > 1 \Rightarrow \mathcal{F}'(k)k - \mathcal{F}(k) = k^{\rho}\left[\mu k^{\mu} + (\rho-1)(k^{\mu}+A)\right] > 0$ *as* $k^{\mu} + A > \bar{k}^{\mu} + A = \delta\bar{k} > 0$. *Thus* $\frac{\mathcal{F}(k)}{k}$ *is an increasing function.*

ii)

$$\mathcal{F}''(k) = \begin{cases} \delta(\rho+1)\rho k^{\rho-1}, & k < \bar{k} \\ A\rho(\rho-1)k^{\rho-2} + (\mu+\rho)(\mu+\rho-1)k^{\mu+\rho-2}, & k \ge \bar{k}. \end{cases}$$

It is clear that $\mathcal{F}''(k) > 0$ for $k < \bar{k}$. There are two cases to be checked when $k \ge \bar{k}$:

$\rho \le 1 \Rightarrow \mathcal{F}''(k) = k^{\mu+\rho-2}\left[(\mu+\rho)(\mu+\rho-1) + A\rho(\rho-1)k^{-\mu}\right] > 0$ *and* $\rho > 1 \Rightarrow \mathcal{F}''(k) = k^{\rho-2}\left[(k^{\mu}+A)(\rho-1)\rho + (\mu+\rho)\mu k^{\mu} + \mu(\rho-1)k^{\mu}\right] > 0$. *Thus, $\mathcal{F}'(k)$ is an increasing function.* ■

Lemma 2 *$\frac{\partial^2 V(k,y)}{\partial k\,\partial y} > 0$ where $V(k,y) = u\left(\mathcal{F}(k) - k\gamma\left(\frac{y-k}{k}\right)\right) + \beta V(y)$, denoting the value function of the social optimization problem, is differentiable.*

Proof. $\frac{\partial^2 V(k,y)}{\partial k\ \partial y} =$

$$-u''\left(\mathcal{F}(k) - k\gamma(\frac{y-k}{k})\right)\gamma'\left(\frac{y-k}{k}\right)\left[\mathcal{F}'(k) - \gamma\left(\frac{y-k}{k}\right) + \gamma'\left(\frac{y-k}{k}\right)\frac{y}{k}\right] + u'\left(\mathcal{F}(k) - k\gamma\left(\frac{y-k}{k}\right)\right)\gamma''\left(\frac{y-k}{k}\right)\frac{y}{k^2}.$$

As γ is a convex function, we know that :

$$-\gamma\left(\frac{y-k}{k}\right) = \gamma(0) - \gamma\left(\frac{y-k}{k}\right) \geq -\gamma'\left(\frac{y-k}{k}\right)\frac{y-k}{k} \Longrightarrow$$
$$\gamma'\left(\frac{y-k}{k}\right)\frac{y}{k} \geq \gamma\left(\frac{y-k}{k}\right) + \gamma'\left(\frac{y-k}{k}\right) > \gamma\left(\frac{y-k}{k}\right).$$

Remembering also that $\mathcal{F}$ is an increasing function and u is a concave function leads us to conclude that $\frac{\partial^2 V(k,y)}{\partial k\ \partial y} > 0$. ■

Lemma 3 *Consider the problem:*

$$\max \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1})$$
$$s.t. \quad x_t \leq x_{t+1} \leq B(x_t)$$

which leads to the value function verifying the Bellman equation given as

$$V(x_0) = \max_{x_0 \leq y \leq B(x_0)} \left[u(x_0, y) + \beta V(y)\right]$$

where B is an increasing function. Let $\{x_t\}$ and $\{x_t'\}$ be optimal paths starting from x_0 and x_0' respectively. If $x_0 < x_0'$ then $x_1 \leq x_1'$.

Proof. Assume on the contrary that $x_0 < x_0'$ and $x_1 > x_1'$. Then we have $x_0 < x_0' \leq x_1' < x_1 \leq B(x_0) < B(x_0')$. By the principle of optimality,

$$\begin{aligned} V(x_0) &= u(x_0, x_1) + \beta V(x_1) \geq u(x_0, x_1') + \beta V(x_1') \\ V(x_0') &= u(x_0', x_1') + \beta V(x_1') \geq u(x_0', x_1) + \beta V(x_1). \end{aligned}$$

By combining the two inequalities we get that:

$$u(x_0, x_1) + u(x_0', x_1') \geq u(x_0, x_1') + u(x_0', x_1)$$

which leads to

$$u(x_0, x_1) - u(x_0, x_1') \geq u(x_0', x_1) - u(x_0', x_1').$$

This inequality can also be written as follows:

$$\int_{x_1'}^{x_1} \frac{\partial V}{\partial y}(x_0, y)\, dy \geq \int_{x_1'}^{x_1} \frac{\partial V}{\partial y}(x_0', y)\, dy.$$

On the other hand, as $x_0 > x_0'$

$$\int_{x_1'}^{x_1} \frac{\partial V}{\partial y}(x_0', y)\, dy > \int_{x_1'}^{x_1} \frac{\partial V}{\partial y}(x_0, y)\, dy$$

leads to a contradiction by means of Lemma 2. ■

Corollary 1 *If $\{k_t\}$ is an optimal path from k_0 then either $k_t \leq k_{t+1}$ for all t or $k_t \geq k_{t+1}$ for all t.*

Proof. *Suppose $k_1 \geq k_0$. By the principle of optimality $\{k_2, k_3, ...\}$ is an optimal path starting from k_1. As $k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right) + k_0$ is an increasing function of k_0, then according to Lemma 3, $k_2 \geq k_1$ and by induction on t, $k_{t+1} \geq k_t$ for $\forall t$. For the case $k_1 \leq k_0$, the proof is the same.* ■

Proposition 1 *If $\{k_t, c_t\}$ is an optimal path from $k_0 > 0$, then $\forall t \geq 0$ $k_t > 0$, $c_t > 0$. Similarly if $k_1 \in \varphi(k_0)$ then $\mathcal{F}(k_0) - k_0\gamma\left(\frac{k_1 - k_0}{k_0}\right) > 0$.*

Proof. *Consider an optimal path $\{c_t, k_t\}$ from k_0 . Obviously the path $\left(\hat{c}_t = 0\right)_{t=0}^{\infty}$ is not optimal. So we may assume that $\hat{c}_0 = 0$ and $\hat{c}_1 > 0$, i.e. :*

$\hat{c}_0 = \mathcal{F}(k_0) - k_0\gamma\left(\frac{k_1 - k_0}{k_0}\right) = 0,$

$\hat{c}_1 = \mathcal{F}(k_1) - k_1\gamma\left(\frac{k_2 - k_1}{k_1}\right) > 0,$

$\hat{c}_t \geq 0,\ t \geq 2.$

Let an alternative feasible path $\tilde{c} = \left\{\tilde{c}_t\right\}$ be defined by:

$\tilde{c}_0 = \mathcal{F}(k_0) - k_0\gamma\left(\frac{(k_1 - \varepsilon) - k_0}{k_0}\right) > 0,$

$\tilde{c}_1 = \mathcal{F}(k_1 - \varepsilon) - (k_1 - \varepsilon)\gamma\left(\frac{k_2 - k_1 + \varepsilon}{k_1 - \varepsilon}\right) > 0,$

$\tilde{c}_t = \hat{c}_t,\ t \geq 2.$

Let $\Theta = \left(u(\tilde{c}_0) - u(\hat{c}_0)\right) + \beta\left(u\left(\tilde{c}_1\right) - u\left(\hat{c}_1\right)\right)$. Then by concavity of u, convexity of γ and Inada condition on u, it is easy to show that $\Theta > 0$ for ε small enough leading to a contradiction with the optimality of $\hat{c}$; thus $k_t > 0$, $c_t > 0$ $\forall t \geq 0$. ■

Proposition 2 *Let $\lambda > 0$ be given.*

1) There exists $k^* > 0$ such that $\forall k_0 < k^*$, $\forall \underset{\sim}{k}$ optimal from k_0 then $k_t = k_0, \forall t$.

2) There exists $k^{**} > 0$ such that $\forall k_0 > k^{**}$, $\forall \underset{\sim}{k}$ optimal from k_0 then $k_t < k_{t+1}, \forall t$ and $k_t \to +\infty$.

Proof. *1) Choose k^* such that $g\left(\frac{\mathcal{F}(k^*)}{k^*}\right) < \bar{g}$ and $\frac{\mathcal{F}_-'(k^*)}{\lambda} + (1 + \alpha) < \frac{1}{\beta}$. We claim that $\forall k_0 < k^*$, $\forall \underset{\sim}{k}$ optimal from k_0 then $k_t = k_0, \forall t$. Assume contrary. Let for $k_0 < k^*$, there exists an increasing sequence $\{k_t\}$ which is optimal and* *satisfies* $k_t \leq k^*, \forall t$. *Then* $k_t \to k_s$ *which satisfies* $\mathcal{F}'(k_s) = \frac{\lambda(1-\beta)}{\beta}$ *due to Euler equation and* $\gamma'(0) = \lambda > 0$. *Since* $\mathcal{F}'$ *is increasing and* $k_s \leq k^* \Rightarrow \mathcal{F}'(k_s) \leq \mathcal{F}'(k^*) < \frac{\lambda(1-\beta)}{\beta}$, *leads to a contraction. Then there exits* t *such that* $k_t > k^*$. *In what follows, let* $T(k_0)$ *denote the first date such that:*

$$\begin{aligned} k_t &< k^* \text{ if } t < T(k_0) \\ k_t &> k^* \text{ if } t \geq T(k_0). \end{aligned}$$

¿From $t = 0$ *to* $T(k_0)$, *due to Euler equation we can write:*

$$u'(c_t)\gamma'\left(\frac{k_{t+1}}{k_t} - 1\right) = \beta u'(c_{t+1}) \left[\mathcal{F}'(k_{t+1}) - \gamma\left(\frac{k_{t+2}}{k_{t+1}} - 1\right) + \gamma'\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right]$$

which reduces to

$$u'(c_t) = \beta u'(c_{t+1}) \left[\frac{\mathcal{F}'(k_{t+1})}{\lambda} + (1+\alpha)\right].$$

Since $\beta\left[\frac{\mathcal{F}'(k)}{\lambda} + (1+\alpha)\right] < 1$, *one has* $c_t > c_{t+1}$. *We have* $c_0 > c_1 > ... > c_{T(k_0)}$. *Now consider a path* $\{k_0^n\}$ *converging to zero. One may assume that* $k^n_{T(k_0^n)}$ *converges to* $\hat{k} \leq k^*$ *and* $k^n_{T(k_0^n)+1}$ *converges to* $\hat{k}' \geq k^*$. *By the upper semi-continuity of* φ, $k^n_{T(k_0^n)+1} \in \varphi(k^n_{T(k_0^n)})$ *leading to that* $\hat{k}' \in \varphi(\hat{k})$. *Since* $\hat{k}' > 0$, *from proposition 1,* $\hat{k} > 0$. *Then following from inada condition we have that* $\mathcal{F}(\hat{k}) - \hat{k}\gamma\left(\frac{\hat{k}' - \hat{k}}{\hat{k}}\right) > 0$ *on the one hand. But* $c^n_{T(k_0^n)} < c_0^n < \mathcal{F}(k_0^n)$. *Then* $c^n_{T(k_0^n)}$ *must converge to zero on the other hand: a contradiction.*

2) Choose k^{**} *such that* $\mathcal{F}'_+(k^{**})\frac{\beta}{1-\beta} \geq \lambda$. *Our claim is that* $\forall k_0 > k^{**}$, $\forall \underset{\sim}{k}$ *optimal from* k_0 *then* $k_t < k_{t+1}, \forall t$ *and* $k_t \to +\infty$. *In order to prove this we will first show that, for any* $k_0 > k^{**}$, *the path* $\tilde{k}_0 = (k_0, k_0, ..., k_0, ...)$ *is not optimal. Then we will show that any optimal path is strictly increasing and no optimal path from* $k_0 > k^{**}$ *converges to a steady state.*

i) Consider the path $\tilde{k}_1 = (k_0, k_0+\varepsilon, k_0+\varepsilon, ..., k_0+\varepsilon...)$ *that is feasible from* $k_0 : k_0 \leq k_0 < k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right) + k_0$. *Since* $k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right) > 0$, *there exists* $\varepsilon > 0$ *such that* $k_0 + \varepsilon \leq k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right) + k_0$. *Then for* $k_1 := k_0 + \varepsilon$ *and* $k_t := k_1$, $\forall t \leq 1$, *we have* $k_0 \leq k_1 \leq k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right) + k_0$ *and* $k_1 \leq k_1 \leq k_1 g\left(\frac{\mathcal{F}(k_1)}{k_1}\right) + k_1$ *that is* $k_t \leq k_{t+1} \leq k_t g\left(\frac{\mathcal{F}(k_t)}{k_t}\right) + k_t$. *Thus there exists* $\varepsilon > 0$ *such that* $\tilde{k}_1$ *is feasible* *from k_0. Now we will show that such a path increases the value of U :*

$$\begin{aligned}
U(\widetilde{k_1}) &= u\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \sum_{t=1}^{\infty}\beta^t u\left(\mathcal{F}(k_0+\varepsilon)\right) \\
&= u\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \frac{\beta}{1-\beta}u\left(\mathcal{F}(k_0+\varepsilon)\right). \\
U(\widetilde{k_0}) &= u\left(\mathcal{F}(k_0)\right) + \frac{\beta}{1-\beta}u\left(\mathcal{F}(k_0)\right).
\end{aligned}$$

Then,

$$\begin{aligned}
U(\widetilde{k_1}) - U(\widetilde{k_0}) = &\left[u\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) - u\left(\mathcal{F}(k_0)\right)\right] \\
&+ \frac{\beta}{1-\beta}\left[u\left(\mathcal{F}(k_0+\varepsilon)\right) - u\left(\mathcal{F}(k_0)\right)\right]
\end{aligned}$$

$$\begin{aligned}
\geq\, & u'\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right)\left(-k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) \\
&+ \frac{\beta}{1-\beta}u'\left(\mathcal{F}(k_0+\varepsilon)\right)\left(\mathcal{F}(k_0+\varepsilon) - \mathcal{F}(k_0)\right)
\end{aligned}$$

$$\geq \varepsilon\left[u'\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right)\left(-\frac{k_0}{\varepsilon}\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \frac{\beta}{1-\beta}u'\left(\mathcal{F}(k_0+\varepsilon)\right)\mathcal{F}'(k_0)\right].$$

Since $\lim_{\varepsilon\to 0}\frac{k_0}{\varepsilon}\gamma\left(\frac{\varepsilon}{k_0}\right) = \gamma'(0) = \lambda$ and $\mathcal{F}'$ is increasing then $\exists k^{**}$ such that for $k_0 > k^{**}$, $\mathcal{F}'(k_0) > \lambda\frac{(1-\beta)}{\beta}$. Thus, there exists $\varepsilon > 0$ such that $U(\widetilde{k_1}) - U(\widetilde{k_0}) > 0$ concluding that $\widetilde{k_0}$ is not optimal.

ii) If there were an optimal path such that $k_1 = k_0$ then as the value function verifies the Bellman equation:

$$k_0 = k_1 \in \underset{k_0\leq y\leq k_0 g\left(\frac{\mathcal{F}(k_0)}{k_0}\right)+k_0}{\arg\max}\left\{u\left(\mathcal{F}(k_0) - k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right) + \beta V(y)\right\}$$

the path $\widetilde{k_0} = (k_0, k_0, .., k_0, ..)$ would then be optimal which is impossible. So necessarily, $k_1 > k_0$. Similarly what follows is that $\forall t$, $k_{t+1} > k_t$.

iii) Writing the Euler equation,

$$\begin{aligned}
&-\gamma'\left(\frac{k_{t+1}-k_t}{k_t}\right)u'\left(\mathcal{F}(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right) + \\
&\quad\beta\left[\mathcal{F}'(k_{t+1}) - \gamma\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right) + \gamma'\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right]\times \\
&\qquad u'\left(\mathcal{F}(k_{t+1}) - k_{t+1}\gamma\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)\right) = 0
\end{aligned}$$

Since k_t is strictly increasing, $\lim_{t\to\infty} k_t$ exists. Suppose on the contrary that k_t converges to $k_s < +\infty$. Then we would have

$$\mathcal{F}'(k_s)\frac{\beta}{1-\beta} = \lambda.$$

But $\mathcal{F}'_+(k^{**})\,\frac{\beta}{1-\beta} \geq \lambda$. A contradiction. So k_t diverges to $+\infty$ inducing that per-capita output grows without bound. ■

Remark 1 *The developing countries for whom the returns to scale are increasing at the early stages of industrialization and the production function becomes concave for higher levels of capital stocks may find or may not find it optimal to industrialize at the early stages of development depending on the quality of the knowledge technology and the production-capital ratio. Even a developed country with a concave production function, according to Proposition 2, if endowed with a low quality of knowledge technology may face a poverty trap with a low level of initial capital stock.*

Remark 2 *The developing countries will be facing the obstacles of low level of initial capital stock, low quality of knowledge technology and as they are facing a production technology which exhibits linear production-capital ratio for low levels of initial capital stock, low level of marginal productivity of capital (δ), in order to be able to take-off . Remember that:*

$\mathcal{F}'(k) = A\rho k^{\rho-1} + (\rho+\mu)\,k^{p+\mu-1},\ k > \bar{k}$ *where* $A = \delta\bar{k} - \bar{k}^{\mu}$. *Then,*

$\frac{dk^{**}}{d\delta} = -\frac{\bar{k}\,\rho k^{**\,(\rho-1)}}{A\rho(\rho-1)k^{**\,(\rho-2)}+(\rho+\mu)(\rho+\mu-1)k^{**\,(\rho+\mu-2)}} < 0.$

That is to say, the countries with higher values of marginal productivity of capital at their early stages of development, will be more prone to take-off.

Theorem 1 *Let $\lambda > 0$ be given. Then there exists k_c such that $\forall k_0 < k_c$, any optimal path $\underset{\sim}{k}$ from k_0 will satisfy $k_t = k_0$, $\forall t$ and $\forall k_0 > k_c$, any optimal path $\underset{\sim}{k}$ from k_0 will satisfy $k_t < k_{t+1}$, $\forall t$, and $k_t \to +\infty$.*

Proof. *Defining $k_c^M = \sup(k^*)$ and $k_c^m = \inf(k^{**})$, our claim is that $k_c^M = k_c^m$.*

i) Assume that $k_c^m < k_c^M$. Take k_0 and k_0' such that $k_c^m < k_0 < k_0' < k_c^M$. From the very definition of k_c^m, $\exists\ \underset{\sim}{k}$ being an optimal path from k_0, $k_t \to +\infty$ whereas from the very definition of k_c^M, $\exists\ \underset{\sim}{k}'$ being an optimal path from k_0', $k_t' = k_0, \forall t$, which is impossible by Lemma 3.

ii) Assume that $k_c^M < k_c^m$. Take k_0 and k_0' such that $k_c^M < k_0 < k_0' < k_c^m$. Similarly, from the very definition of k_c^M, $\exists\ \underset{\sim}{k}$ being an optimal path from k_0, $k_t \to +\infty$ and from the very definition of k_c^m, $\exists\ \underset{\sim}{k}'$ being an optimal path from k_0', $k_t' = k_0, \forall t$, which is impossible by Lemma 3.

Thus, combining the two cases suffices to prove that $k_c^M = k_c^m$. ■

Remark 3 *Note that, for a given level of quality of knowledge technology, for a poor country with a low level of initial capital stock it could be optimal not to take-off while a rich contry with a higher level of initial capital stock than the critical level could have an unbounded growth and both countries have the same technology.*

3 Conclusion

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References

[1] Askenazy, P., and C. Le Van, 1999, A model of optimal growth strategy, Journal of Economic Theory 85(1), 24-51.

[2] Cass, D., 1965, Optimal growth in an aggregative model of capital accumulation, Rewiev of Economic Studies 32, 233-240.

[3] Dechert, W.D. and K. Nishimura, 1983, A complete characterization of optimal growth paths in an aggregated model with non-concave production function, Journal of Economic Theory 31, 332-354.

[4] Dimaria, C., C. Le Van, 1999, Debt, corruption, R&D and growth in developing countries, Macroeconomic Dynamics, forthcoming.

[5] Koopmans, T.C., 1965, On the concept of optimal economic growth, The Econometric Approach to Development Planning, edited by North Holland.

[6] Le Van, C., Morhaim, L., and C. Dimaria, 2000, The discrete time version of the Romer model, Economic Theory, forthcoming.

[7] Majumdar, M., and T. Mitra, 1982, Intertemporal Allocation with a non-convex technology: The aggregative framework, Journal of Economic Theory 27, 101-136.

[8] Romer, P., 1986, Increasing returns and long-run growth, Journal of Political Economy 94, 1002-1037.